Difference of sums containing products of binomial coefficients and their logarithms

Allen R. Miller
1616 Eighteenth Street NW
Washington, DC 20009

Ira S. Moskowitz[1]
Center for High Assurance Computer Systems
Naval Research Laboratory
Washington, DC 20375

Abstract

Properties of the difference of two sums containing products of binomial coefficients and their logarithms which arise in the application of Shannon's information theory to a certain class of covert channels are deduced. Some allied consequences of the latter are also recorded.

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[1] *E-mail address*: moskowitz@itd.nrl.navy.mil

1. Introduction

Traffic analysis in anonymizing network configurations can be used to open a covert communication channel. Recently, Moskowitz et al. [2] have developed a mathematical model (whose salient feature is given by equation (1.1) below) that describes the situation of oneway messages passing from private Enclave_1 to private Enclave_2 in which each communication is encrypted and passes through exit and entry mix firewalls. Furthermore, there are $n+1$ senders in Enclave_1, and one of them called Alice is malicious. The other n clueless transmitters are benign. Every sender may send at most one message per unit time t to Enclave_2. All messages from Enclave_1 to Enclave_2 pass through a public line that is subject to eavesdropping by an eavesdropper called Eve who knows the value of n. The only action Eve can take is to count the number of messages per t going from Enclave_1 to Enclave_2. The n clueless transmitters all send their messages per t as independently and identically distributed Bernoulli random variables with parameter q. Alice acts independently (through ignorance of the n clueless senders) when deciding to send a message. Thus, Alice by sending or not sending a message affects the number of messages that Eve counts. This in brief is the covert channel.

A normalizing noise term $S(n)$ is defined corresponding to the degree of anonymity afforded by the n clueless senders transmitting as fair coins ($q = 1/2$) which is the situation when maximal anonymity occurs. For $n = 1, 2, 3, \ldots$ $S(n)$ is shown in [2] to be given by

$$S(n) = \frac{1}{2^n} \sum_{k=0}^{n} \left[\frac{1}{2} \binom{n+1}{k} \ln \binom{n+1}{k} - \binom{n}{k} \ln \binom{n}{k} \right] , \tag{1.1}$$

and by employing Shannon's information theory the capacity is given by $1 - S(n)/\ln 2$. Moreover, an important property of this model is that the capacity should decrease monotonically to zero with increasing n.

For brevity in the sequel we define $T(n) \equiv 2^n S(n)$. The expression for $T(n)$ is of interest in its own right, since it is the difference of two divergent sums that contain products of binomial coefficients and their logarithms. Moreover, these types of sums appear not to be readily found in the mathematical literature.

In the present investigation we shall obtain in section 2 a simpler algebraic representation for $T(n)$ which contains the sum $\sum_{k=0}^{n} \binom{n}{k} \ln(k+1)$ that is similar, for example, to a particular sum involved in a representation for $\ln, (x+1)$ deduced by Stirling for $x > -1$ (see [1, equation (5.47)]). We shall also derive two integral representations and two representations involving integrals for $S(n)$. Then in section 3 we give a detailed proof that $S(n)$ is monotonically increasing with increasing n. This property of $S(n)$ alluded to above was verified numerically in [2] for $n \leq 7750$. Finally, in section 4, we show that $S(n) \to \ln 2$ as $n \to \infty$. We record some consequences of the latter as well.

2. Reduction of $S(n)$ to a simpler sum

Recall that $T(n) = 2^n S(n)$ where $S(n)$ is given by equation (1.1). We shall show below that

$$T(n) = 2^n \ln(n+1) - \sum_{k=0}^{n} \binom{n}{k} \ln(k+1) \tag{2.1}$$

and so $S(n)$ may be written elegantly as

$$S(n) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \ln\left(\frac{n+1}{k+1}\right) \ . \tag{2.2}$$

To prove the latter we define

$$\alpha(n) \equiv \sum_{k=0}^{n} \binom{n}{k} \ln \binom{n}{k} \ ,$$

where the binomial coefficient $\binom{n}{k} = n!/k!(n-k)!$ which we agree vanishes if $k > n$ or if $k < 0$. Thus

$$\alpha(n) = \ln n! \sum_{k=0}^{n} \binom{n}{k} - \sum_{k=0}^{n} \binom{n}{k} \ln k! - \sum_{k=0}^{n} \binom{n}{k} \ln(n-k)! \ . \tag{2.3}$$

The first sum equals 2^n. By noting that

$$\binom{n}{k} = \binom{n}{n-k} \tag{2.4}$$

and reversing the order of summation (i.e. by letting $k \mapsto n-k$) in the third sum, we see that the second and third sums in equation (2.3) are equal so that

$$\alpha(n) = 2^n \ln n! - 2\sum_{k=0}^{n} \binom{n}{k} \ln k! \tag{2.5a}$$

and so

$$\alpha(n+1) = 2^{n+1} \ln(n+1)! - 2\sum_{k=0}^{n+1} \binom{n+1}{k} \ln k! \ . \tag{2.5b}$$

From equation (1.1) and the definitions of $T(n)$ and $\alpha(n)$ it is evident that

$$T(n) = \frac{1}{2}\alpha(n+1) - \alpha(n)$$

and so by using equations (2.5) we have

$$T(n) = 2^n \ln(n+1) + \sum_{k=0}^{n+1} \left[2\binom{n}{k} - \binom{n+1}{k} \right] \ln k! \ . \tag{2.6}$$

Now observing that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{2.7}$$

equation (2.6) gives

$$T(n) = 2^n \ln(n+1) + \sum_{k=0}^{n} \binom{n}{k} \ln k! - \sum_{k=1}^{n+1} \binom{n}{k-1} \ln k! \ .$$

Finally, since

$$\sum_{k=1}^{n+1} \binom{n}{k-1} \ln k! = \sum_{k=0}^{n} \binom{n}{k} \ln(k+1)!$$

we obtain equation (2.1).

Thus also

$$S(n) = \ln(n+1) - \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \ln(k+1) \tag{2.8}$$

which shows that $-S(n)$ is the difference of a divergent weighted sum of logarithms $\ln(k+1)$ (since $\sum_{k=0}^{n} 2^{-n} \binom{n}{k} = 1$) and the montonically increasing logarithm $\ln(n+1)$. This observation brings to mind and is analogous to one of the many formulations for Euler's constant γ (cf. e.g. [1, equation (6.65)]) which is the limiting value as $n \to \infty$ of the much simpler difference of the divergent harmonic series $\sum_{k=1}^{n} 1/k$ and $\ln(n+1)$. (However, we show in section 4 that $S(n) \to \ln 2$ as $n \to \infty$.) We have already noted that series of the type appearing in equation (2.1) appear in the literature in other contexts.

We conclude this section by deriving four integral representations for $S(n)$. We start with (see [3, section 2.6.17., equation (1)])

$$\ln\left(\frac{n+1}{k+1}\right) = \int_0^1 (x^n - x^k) \frac{dx}{\ln x} \ , \tag{2.9}$$

where $n > -1$ and $k > -1$. Multiplying both sides of equation (2.9) by the binomial coefficient $\binom{n}{k}$ and summing over the index k gives

$$\sum_{k=0}^{n} \binom{n}{k} \ln\left(\frac{n+1}{k+1}\right) = \int_0^1 [2^n x^n - (1+x)^n] \frac{dx}{\ln x} \ .$$

Dividing each side of the latter equation by 2^n and noting equation (2.2) we deduce for $n = 1, 2, 3 \ldots$

$$S(n) = \int_0^1 \left[x^n - \left(\frac{1+x}{2}\right)^n\right] \frac{dx}{\ln x} \ . \tag{2.10a}$$

Now making the transformation $x = e^{-t}$ in the latter integral yields

$$S(n) = \int_0^\infty e^{-(n+1)t} \left[\left(\frac{1+e^t}{2}\right)^n - 1\right] \frac{dt}{t} \ . \tag{2.10b}$$

However, letting $n = 0$ in equation (2.9) we may start with

$$\ln(k+1) = \int_0^1 (x^k - 1) \frac{dx}{\ln x} \ .$$

Multiplying the latter equation by $\begin{pmatrix} n \\ k \end{pmatrix}$ and summing over k now gives

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ln(k+1) = \int_0^1 [(1+x)^n - 2^n] \frac{dx}{\ln x}$$

which upon dividing by 2^n and noting equation (2.8) yields

$$S(n) = \ln(n+1) + \int_0^1 \left[1 - \left(\frac{1+x}{2}\right)^n\right] \frac{dx}{\ln x} \ .$$

Upon making the transformation $x = e^{-t}$ in the integral this result can also be written as

$$S(n) = \ln(n+1) + \int_0^\infty \left[\left(\frac{1+e^{-t}}{2}\right)^n - 1\right] \frac{e^{-t}}{t} dt \ .$$

3. Monotonicity and boundedness of $S(n)$

We now prove that $S(n)$ is increasing with increasing $n \geq 1$, i.e. $S(n+1) > S(n)$ for $n = 1, 2, 3, \ldots$.

By employing equation (2.8) we have

$$2^{n+1}[S(n+1) - S(n)] = 2^{n+1} \ln\left(\frac{n+2}{n+1}\right) + 2\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ln(k+1) - \sum_{k=0}^{n+1} \begin{pmatrix} n+1 \\ k \end{pmatrix} \ln(k+1) \ . \tag{3.1a}$$

Now utilizing equation (2.7)

$$\sum_{k=0}^{n+1} \begin{pmatrix} n+1 \\ k \end{pmatrix} \ln(k+1) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ln(k+1) + \sum_{k=1}^{n+1} \begin{pmatrix} n \\ k-1 \end{pmatrix} \ln(k+1) \tag{3.1b}$$

and noting that

$$\sum_{k=1}^{n+1} \begin{pmatrix} n \\ k-1 \end{pmatrix} \ln(k+1) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ln(k+2) \tag{3.1c}$$

we have from equations (3.1)

$$2^{n+1}[S(n+1)-S(n)] = 2^{n+1}\ln\left(\frac{n+2}{n+1}\right) - \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\ln\left(\frac{k+2}{k+1}\right) \ . \tag{3.2}$$

There are two cases to consider: n odd and n even. Accordingly, letting $n \mapsto 2n$ in equation (3.2) gives for $n = 1, 2, 3, \ldots$

$$2^{2n+1}[S(2n+1)-S(2n)] = 2^{2n+1}\ln\left(\frac{2n+2}{2n+1}\right) - \sum_{k=0}^{2n}\left(\begin{array}{c} 2n \\ k \end{array}\right)\ln\left(\frac{k+2}{k+1}\right) \ .$$

Since $\left(\begin{array}{c} 2n \\ k \end{array}\right) \leq \left(\begin{array}{c} 2n \\ n \end{array}\right)$ where equality holds if and only if $k = n$ we have

$$2^{2n+1}[S(2n+1)-S(2n)] > 2^{2n+1}\ln\left(\frac{2n+2}{2n+1}\right) - \left(\begin{array}{c} 2n \\ n \end{array}\right)\sum_{k=0}^{2n}\ln\left(\frac{k+2}{k+1}\right) \ . \tag{3.3}$$

The sum of logarithms in the latter inequality is equal to $\ln(\frac{n+1}{n})$ and it is easy to show that

$$\left(\begin{array}{c} 2n \\ n \end{array}\right) = \frac{2^{2n}}{\sqrt{\pi}}\ \frac{,\ (\frac{1}{2}+n)}{,\ (1+n)} \ .$$

Thus defining for $n = 1, 2, 3, \ldots$

$$\beta(n) \equiv \frac{1}{2\sqrt{\pi}}\ \frac{,\ (\frac{1}{2}+n)}{,\ (1+n)} \tag{3.4}$$

we can write inequality (3.3) as

$$S(2n+1)-S(2n) > \ln\left(\frac{2n+2}{2n+1}\right) - \beta(n)\ln\left(\frac{n+1}{n}\right) \ . \tag{3.5}$$

Now $\beta(1) = 1/4$ and it is easy to show that $\beta(n+1) < \beta(n)$. Thus equation (3.5) gives

$$\begin{aligned} S(2n+1)-S(2n) &> \ln\left(\frac{2n+2}{2n+1}\right) - \frac{1}{4}\ln\left(\frac{n+1}{n}\right) \\ &= \ln\left[\left(\frac{n+1}{n+1/2}\right)\left(\frac{n}{n+1}\right)^{1/4}\right] \end{aligned}$$

and so

$$S(2n+1) - S(2n) > \frac{1}{4} \ln \left[\frac{n(n+1)^3}{(n+1/2)^4} \right] .$$

It is readily shown for $n \geq 1$ that $n(n+1)^3/(n+1/2)^4 > 1$ and so $S(2n+1) > S(2n)$.

Next, letting $n \mapsto 2n+1$ in equation (3.2) gives for $n = 0, 1, 2, \ldots$

$$2^{2n+2}[S(2n+2) - S(2n+1)] = 2^{2n+2} \ln \left(\frac{2n+3}{2n+2} \right) - \sum_{k=0}^{2n+1} \begin{pmatrix} 2n+1 \\ k \end{pmatrix} \ln \left(\frac{k+2}{k+1} \right) .$$

Since $\begin{pmatrix} 2n+1 \\ k \end{pmatrix} \leq \begin{pmatrix} 2n+1 \\ n \end{pmatrix} = \begin{pmatrix} 2n+1 \\ n+1 \end{pmatrix}$ where equality holds if and only if either $k = n$ or $k = n+1$ we have

$$2^{2n+2}[S(2n+2) - S(2n+1)] \geq 2^{2n+2} \ln \left(\frac{2n+3}{2n+2} \right) - \begin{pmatrix} 2n+1 \\ n \end{pmatrix} \sum_{k=0}^{2n+1} \ln \left(\frac{k+2}{k+1} \right) , \quad (3.6)$$

where equality holds if and only if $n = 0$. The sum of logarithms in the latter inequality is equal to $\ln(\frac{2n+3}{2n+1})$ and it is easy to show that

$$\begin{pmatrix} 2n+1 \\ n \end{pmatrix} = \frac{2^{2n+1}}{\sqrt{\pi}} \frac{,(\frac{3}{2}+n)}{,(2+n)}$$

so that noting equation (3.4)

$$\begin{pmatrix} 2n+1 \\ n \end{pmatrix} = 2^{2n+2} \beta(n+1) .$$

Thus we obtain from equation (3.6)

$$S(2n+2) - S(2n+1) \geq \ln \left(\frac{2n+3}{2n+2} \right) - \beta(n+1) \ln \left(\frac{2n+3}{2n+1} \right) ,$$

where $n = 0, 1, 2, \ldots$. Since $\beta(1) = 1/4$ and $\beta(n+1)$ decreases with increasing n

$$\begin{aligned} S(2n+2) - S(2n+1) &\geq \ln \left(\frac{2n+3}{2n+2} \right) - \frac{1}{4} \ln \left(\frac{2n+3}{2n+1} \right) \\ &= \frac{1}{4} \ln \left[\frac{(2n+1)(2n+3)^3}{(2n+2)^4} \right] . \end{aligned}$$

Finally, setting $m = 2n+1$ is is easy to show that $m(m+2)^3/(m+1)^4 > 1$ for $m \geq 1$ and so $S(2n+2) > S(2n+1)$ which completes the proof of the monotonicity of $S(n)$.

We conclude this section by showing for $n \geq 1$ that

$$S(n) < 2(1 - 2^{-n-1}) < 2 \ . \tag{3.7}$$

Assuming the latter, since $S(n)$ is monotonically increasing and bounded we may define

$$\mu \equiv \lim_{n\to\infty} S(n) \ , \tag{3.8}$$

where from inequalities (3.7) we must have $0 < \mu < 2$.

To show inequalities (3.7), upon observing that

$$\ln\left(\frac{n+1}{k+1}\right) < \frac{n+1}{k+1} \qquad (k = 0, 1, \ldots, n) \ ,$$

we have from equation (2.2)

$$S(n) < \frac{n+1}{2^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{k+1} \ .$$

Since (see e.g. [3, section 4.2.2., equation (42)])

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{k+1} = \frac{1}{n+1}(2^{n+1} - 1) \ ,$$

it is evident that $S(n) < 2 - 2^{-n}$ and we are done.

4. The limiting value of $S(n)$ as $n \to \infty$

From equation (2.2) we may write

$$S(n) = -\frac{1}{2^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ln\left(1 - \frac{n-k}{n+1}\right) \ .$$

Letting $k \mapsto n-k$ and using equation (2.4) in this result then yields

$$S(n) = -\frac{1}{2^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ln\left(1 - \frac{k}{n+1}\right) \ . \tag{4.1}$$

Since for $0 \leq z < 1$

$$\ln(1-z) = -\sum_{p=1}^{\infty} \frac{z^p}{p} \tag{4.2}$$

setting $z = k/(n+1)$ we have from equations (4.1) and (4.2)

$$S(n) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sum_{p=1}^{\infty} \frac{1}{p} \left(\frac{k}{n+1} \right)^p \ .$$

Formally interchanging the order of summation then gives

$$S(n) = \frac{1}{2^n} \sum_{p=1}^{\infty} \frac{1}{p} \frac{1}{(n+1)^p} \sum_{k=1}^{n} \binom{n}{k} k^p \ , \tag{4.3}$$

where for integers $p > 0$

$$\sum_{k=1}^{n} \binom{n}{k} k^p = 2^{n-p} \binom{n}{p} p! + 2^n \sum_{k=1}^{p-1} \binom{n}{k} \left(-\frac{1}{2}\right)^k \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^p \ , \tag{4.4}$$

and it is understood that the latter second term vanishes when $p = 1$. Equation (4.4) is derived *ab initio* by Schwatt (cf. [4, p. 84, equation (34)]) where here a misprint is now corrected.

Since $\binom{n}{p} = n!/p!(n-p)!$ we have from equations (4.3) and (4.4)

$$S(n) = \sum_{p=1}^{\infty} \frac{(1/2)^p}{p} \frac{M_p(n)}{(n+1)^p} + \sum_{p=1}^{\infty} \frac{1}{p} \frac{Q_{p-1}(n)}{(n+1)^p} \ , \tag{4.5}$$

where we have defined

$$M_p(n) \equiv \frac{n!}{(n-p)!} \tag{4.6}$$

and

$$Q_{p-1}(n) \equiv \sum_{k=1}^{p-1} \binom{n}{k} \left(-\frac{1}{2}\right)^k \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^p \ . \tag{4.7}$$

Observing that the first and second p-summations in equation (4.5) will terminate respectively when $p > n$ and $p - 1 > n$, this result can only be valid as a divergent asymptotic expansion for $S(n)$ about $n = \infty$.

However, it is easily seen from equation (4.6) that

$$M_p(n) = n(n-1)\dots(n-(p-1))$$

and so $M_p(n)$ is a monic polynomial in n of degree p. Thus for each $p > 0$ it is evident that

$$\frac{M_p(n)}{(n+1)^p} = 1 + O\left(\frac{1}{n}\right) \qquad (n \to \infty) \ . \tag{4.8a}$$

Moreover, from equation (4.7) it is also easily seen that $Q_{p-1}(n)$ is a polynomial in n of degree $p-1$. Let c_{p-1} be the coefficient of n^{p-1} in $Q_{p-1}(n)$. Then also from equation (4.7) we have

$$c_{p-1} = \frac{(-1/2)^{p-1}}{(p-1)!} \sum_{\ell=0}^{p-1} (-1)^\ell \begin{pmatrix} p-1 \\ \ell \end{pmatrix} \ell^p \ .$$

Schwatt [4, p. 101, equation (196)] has shown for $p \geq 1$ that

$$\sum_{\ell=0}^{p-1} (-1)^\ell \begin{pmatrix} p-1 \\ \ell \end{pmatrix} \ell^p = (-1)^{p-1} \frac{1}{2}(p-1)p!$$

so that

$$c_{p-1} = \left(\frac{1}{2}\right)^p p(p-1) \ .$$

Thus we have

$$\frac{Q_{p-1}(n)}{(n+1)^p} = \left(\frac{1}{2}\right)^p p(p-1) O\left(\frac{1}{n}\right) \qquad (n \to \infty) \tag{4.8b}$$

and by using equations (4.8) we have from equation (4.5) the asymptotic result

$$S(n) = \left[1 + O\left(\frac{1}{n}\right)\right] \sum_{p=1}^{\infty} \frac{(1/2)^p}{p} + O\left(\frac{1}{n}\right) \sum_{p=1}^{\infty} \left(\frac{1}{2}\right)^p (p-1) \qquad (n \to \infty) \ . \tag{4.9}$$

Recalling equation (4.2) the first sum in equation (4.9) equals $\ln 2$ and it is easily shown that the second sum reduces to unity so that we can now write equation (4.9) as

$$S(n) = \left[1 + O\left(\frac{1}{n}\right)\right] \ln 2 + O\left(\frac{1}{n}\right) \qquad (n \to \infty) \ .$$

Finally, since we have already shown in section 3 that $\lim_{n\to\infty} S(n)$ exists as a positive number $\mu < 2$, it is evident from the latter result that $\mu = \ln 2$.

We end by recording two interesting corollaries that are obtained immediately from equations (2.10), namely

$$\lim_{n\to\infty} \int_0^1 \left[x^n - \left(\frac{1+x}{2} \right)^n \right] \frac{dx}{\ln x} = \ln 2$$

and

$$\lim_{n\to\infty} \int_0^1 e^{-(n+1)x} \left[\left(\frac{1+e^x}{2} \right)^n - 1 \right] \frac{dx}{x} = \ln 2 \ .$$

5. Conclusions

We have discussed how an important application in anonymity and covert channels leads to the problem of deducing properties of the difference of sums containing products of binomial coefficients and their logarithms. Intuitively, the capacity of the covert channel should decrease monotonically to zero as the number of transmitters increases, and the analysis provided herein proves this.

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