REDUCTION OF A CLASS OF FOX-WRIGHT PSI FUNCTIONS FOR CERTAIN RATIONAL PARAMETERS[1]

by

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Abstract

The Fox-Wright Psi function is a special case of Fox's H-function and a generalization of the generalized hypergeometric function. In the present paper we show that the Psi function reduces to a single generalized hypergeometric function when certain of its parameters are integers and to a finite sum of generalized hypergeometric functions when these parameters are rational numbers. Applications to the solution of algebraic trinomial equations and to a problem in information theory are provided. A connection with Meijer's G-function is also discussed.

KEYWORDS: Fox-Wright Psi function, zeros of trinomials, special functions, information theory.

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1. INTRODUCTION

The Fox-Wright Psi function ${}_p\Psi_q[z]$ and its normalization ${}_p\Psi_q^*[z]$ are hypergeometric functions whose series representations are given by

$$
{}_p\Psi_q\left[\begin{array}{l}(\alpha_1, A_1), \ldots, (\alpha_p, A_p);\\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} \Gamma(\alpha_k + A_k n)}{\prod_{k=1}^{q} \Gamma(\beta_k + B_k n)} \frac{z^n}{n!}
$$

$$
{}_p\Psi_q^*\left[\begin{array}{l}(\alpha_1, A_1), \ldots, (\alpha_p, A_p);\\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{array} z\right] = \frac{\Gamma(\beta_1) \ldots \Gamma(\beta_q)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_q)} {}_p\Psi_q\left[\begin{array}{l}(\alpha_1, A_1), \ldots, (\alpha_p, A_p);\\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{array} z\right] \tag{1.1}
$$

where $\Gamma(z)$ is the Gamma function. Thus ${}_p\Psi_q[z]$ is a special case of Fox's H-function $H_{k,\ell}^{m,n}[z]$ (see e.g. [1, p. 50]) and ${}_p\Psi_q^*[z]$ is a generalization of the familiar generalized hypergeometric function ${}_pF_q[z]$:

$$
{}_pF_q\left[\begin{array}{l}\alpha_1, \ldots, \alpha_p;\\ \beta_1, \ldots, \beta_q;\end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \tag{1.2}
$$

where the Pochhammer or shifted factorial symbol is defined by $(a)_n = \Gamma(a+n)/\Gamma(a)$ for non-negative integers n. Clearly, if we set $A_i = 1$ $(i = 1, \ldots, p)$, $B_i = 1$ $(i = 1, \ldots, q)$ in Equation (1.1), ${}_p\Psi_q^*[z]$ reduces to ${}_pF_q[z]$ given by Equation (1.2).

In what follows we shall consider only the special case of the Fox-Wright function where $p = q = 1$; thus

$$
{}_1\Psi_1^*\left[\begin{array}{l}(\alpha, A);\\ (\beta, B);\end{array} z\right] = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + An)}{\Gamma(\beta + Bn)} \frac{z^n}{n!} \ . \tag{1.3}
$$

The importance of ${}_1\Psi_1^*[z]$ has recently been indicated by its connection with elementary number theory via Fermat's last theorem [2],[3] and in applied problems via the solution of trinomial equations [4]. Thus it appears worthwhile studying and recording further properties of the Psi function. To this end in the present paper we shall show that ${}_1\Psi_1^*[z]$ reduces to a single generalized hypergeometric function when A and

B assume integer values and that when these same parameters are rational numbers, ${}_1\Psi_1^*[z]$ reduces to a finite sum of generalized hypergeometric functions. Then in section 4 we shall give further applications relevant to the solution of trinomials in general and in particular in section 6 to certain trinomials which arise in a problem in information theory. In section 5 we give a reduction of a particular case of Meijer's G-function.

2. THE PSI FUNCTION FOR INTEGER A AND B

Assume that A and B in Equation (1.3) are positive integers. Clearly, for $A = B = 1$

$$ {}_1\Psi_1^* \left[\begin{array}{c} (\alpha, 1); \\ (\beta, 1); \end{array} z \right] = {}_1F_1 \left[\begin{array}{c} \alpha; \\ \beta; \end{array} z \right] \tag{2.1} $$

so that in this case we have reduction to the confluent hypergeometric function ${}_1F_1[z]$ which converges for all z in the finite complex plane.

By using Gauss's multiplication theorem for the Gamma function it follows that

$$ \Gamma[k(n+\mu)] = \Gamma(k\mu)(k^k)^n (\mu)_n \left(\mu + \frac{1}{k}\right)_n \cdots \left(\mu + \frac{k-1}{k}\right)_n \tag{2.2} $$

for positive integers k and non-negative integers n. Thus since $\Gamma(\alpha + An) = \Gamma[A(n + \alpha/A)]$ we obtain (cf. [5, p. 240, Eq. (I.26)])

$$ \Gamma(\alpha + An) = \Gamma(\alpha) \left(\frac{\alpha}{A}\right)_n \left(\frac{\alpha+1}{A}\right)_n \cdots \left(\frac{\alpha + A - 1}{A}\right)_n (A^A)^n \tag{2.3} $$

which when used together with Equation (1.3) yields

$$ {}_1\Psi_1^* \left[\begin{array}{c} (\alpha, A); \\ (\beta, B); \end{array} z \right] = {}_AF_B \left[\begin{array}{c} \frac{\alpha}{A}, \ldots, \frac{\alpha+A-1}{A}\ ; \\ \frac{\beta}{B}, \ldots, \frac{\beta+B-1}{B}\ ; \end{array} \frac{A^A z}{B^B} \right] \tag{2.4} $$

for positive integers A and B. Obviously Equation (2.4) reduces to Equation (2.1) when $A = B = 1$.

Similarly, for integer $A \geq 1$

$$ \Gamma(\alpha - An) = \frac{\Gamma(\alpha)}{(\frac{1-\alpha}{A})_n (\frac{2-\alpha}{A})_n \cdots (\frac{A-\alpha}{A})_n [(-A)^A]^n} \tag{2.5} $$

which follows from Equation (2.3) and the identity $\Gamma(\alpha - n)/\Gamma(\alpha) = (-1)^n/(1-\alpha)_n$ with n replaced by An. Thus we have also from Equations (2.3), (2.5) and (1.3) for A and B positive integers the following:

$$ {}_1\Psi_1^* \left[\begin{array}{c} (\alpha, -A); \\ (\beta, -B); \end{array} z \right] = {}_BF_A \left[\begin{array}{c} \frac{1-\beta}{B}, \frac{2-\beta}{B}, \ldots, \frac{B-\beta}{B}; \\ \frac{1-\alpha}{A}, \frac{2-\alpha}{A}, \ldots, \frac{A-\alpha}{A}; \end{array} \frac{(-B)^B z}{(-A)^A} \right] \tag{2.6} $$

$$
{}_1\Psi_1^*\left[\begin{matrix}(\alpha,-A) & ; \\ (\beta,B) & ;\end{matrix}\; z\right] = {}_0F_{A+B}\left[\frac{1-\alpha}{A},\ldots,\frac{A-\alpha}{A},\frac{\beta}{B},\ldots,\frac{\beta+B-1}{B};\frac{z}{(-A)^A B^B}\right] \tag{2.7}
$$

$$
{}_1\Psi_1^*\left[\begin{matrix}(\alpha,A) & ; \\ (\beta,-B) & ;\end{matrix}\; z\right] = {}_{A+B}F_0\left[\frac{\alpha}{A},\ldots,\frac{\alpha+A-1}{A},\frac{1-\beta}{B},\ldots,\frac{B-\beta}{B};A^A(-B)^B z\right]. \tag{2.8}
$$

Note that the right hand side of Equation (2.8) diverges for $z \neq 0$ except when it is a polynomial in which case it converges for all z. In addition, Equations (2.4), (2.6)-(2.8) are valid for either $A = 0$ or $B = 0$ by deleting the parameters which contain them.

3. THE PSI FUNCTION FOR RATIONAL A AND B

Assume now that A and B are positive rational numbers. We may always assume that $A = a/k$ and $B = b/k$ where the integer pairs a, k and b, k need not be relatively prime. Thus

$$
\begin{aligned}
{}_1\Psi_1^*\left[\begin{array}{c}(\alpha, a/k);\\ (\beta, b/k);\end{array} z\right] &= \frac{,(\beta)}{,(\alpha)}\sum_{n=0}^{\infty}\frac{,(\alpha+\frac{a}{k}n)}{,(\beta+\frac{b}{k}n)}\,\frac{z^n}{n!}\\
&= 1+\frac{,(\beta)}{,(\alpha)}\sum_{n=1}^{\infty}\frac{,(\alpha+\frac{a}{k}n)}{,(\beta+\frac{b}{k}n)}\,\frac{z^n}{n!}\;. \qquad (3.1)
\end{aligned}
$$

Now partition the positive integers n according to $n = kp - q$ where $q = 0, 1, \ldots, k-1$ and $p = 1, 2, 3, \ldots$. Calling for the moment the Psi function in Equation (3.1) S we obtain

$$
\begin{aligned}
S &= 1+\frac{,(\beta)}{,(\alpha)}\sum_{q=0}^{k-1}\sum_{p=1}^{\infty}\frac{,[\alpha+a(p-q/k)]}{,[\beta+b(p-q/k)]}\,\frac{z^{kp-q}}{,(1+kp-q)}\\
&= 1+\frac{,(\beta)}{,(\alpha)}\sum_{q=0}^{k-1} z^{k-q}\sum_{p=0}^{\infty}\frac{(z^k)^p}{,(1+k-q+kp)}\,\frac{,[a(p+\frac{k-q}{k}+\frac{\alpha}{a})]}{,[b(p+\frac{k-q}{k}+\frac{\beta}{b})]}\\
&= 1+\frac{,(\beta)}{,(\alpha)}\sum_{r=1}^{k} z^{r}\sum_{p=0}^{\infty}\frac{(z^k)^p}{,[k(p+\frac{r+1}{k})]}\,\frac{,[a(p+\frac{r}{k}+\frac{\alpha}{a})]}{,[b(p+\frac{r}{k}+\frac{\beta}{b})]}
\end{aligned}
\qquad (3.2)
$$

where the latter result was obtained by reversing the order of summation by setting $r = k - q$ in the penultimate sum.

Referring to Equation (3.2) set

$$
\mu_1 = \frac{r}{k}+\frac{\alpha}{a}\;, \quad \mu_2 = \frac{r+1}{k}\;, \quad \mu_3 = \frac{r}{k}+\frac{\beta}{b}\;.
$$

Then by using Equation (2.2) we have

$$
\begin{aligned}
\frac{,[a(p+\mu_1)]}{,[k(p+\mu_2)],\,[b(p+\mu_3)]} &= \left(\frac{a^a}{k^k b^b}\right)^p \frac{,(a\mu_1)}{,(k\mu_2),\,(b\mu_3)}\\
&\cdot \frac{(\mu_1)_p(\mu_1+\frac{1}{a})_p\ldots(\mu_1+\frac{a-1}{a})_p}{(\mu_2)_p(\mu_2+\frac{1}{k})_p\ldots(\mu_2+\frac{k-1}{k})_p(\mu_3)_p(\mu_3+\frac{1}{b})_p\ldots(\mu_3+\frac{b-1}{b})_p}
\end{aligned}
$$

which we use together with Equation (3.2) to obtain for $k = 1, 2, 3, \ldots$

$$
{}_1\Psi_1^*\left[\begin{array}{l}(\alpha, a/k);\\ (\beta, b/k);\end{array}\ z\right] = 1 + \frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=1}^{k}\frac{\Gamma(\alpha+\frac{a}{k}r)}{\Gamma(\beta+\frac{b}{k}r)}\ \frac{z^r}{r!}
$$

$$
\cdot\,{}_{a+1}F_{b+k}\left[\begin{array}{ll}1, \frac{r}{k}+\frac{\alpha}{a}, \frac{r}{k}+\frac{\alpha+1}{a}, \ldots, \frac{r}{k}+\frac{\alpha+a-1}{a} & ;\ \frac{a^a}{b^b}\left(\frac{z}{k}\right)^k\\ \frac{r}{k}+\frac{\beta}{b}, \frac{r}{k}+\frac{\beta+1}{b}, \ldots, \frac{r}{k}+\frac{\beta+b-1}{b}, \frac{r+1}{k}, \frac{r+2}{k}, \ldots, \frac{r+k}{k} & ;\end{array}\right] \tag{3.3}
$$

where a and b are non-negative integers, the result being valid for either $a = 0$ or $b = 0$ by deleting the parameters which contain them.

Next, from Equation (2.5) since $\Gamma(\alpha - An) = \Gamma[-A(n - \alpha/A)]$, it easily follows that

$$
\Gamma[-k(n+\mu)] = \frac{\Gamma(-\mu k)}{(\mu+\frac{1}{k})_n(\mu+\frac{2}{k})_n\ldots(\mu+\frac{k}{k})_n[(-k)^k]^n} \tag{3.4}
$$

where k is a positive integer and n is a non-negative integer. Equations (2.2) and (3.4) may then be used together with Equation (3.2) to obtain for non-negative integers a and b the following results for $k = 1, 2, 3, \ldots$:

$$
{}_1\Psi_1^*\left[\begin{array}{l}(\alpha, -a/k);\\ (\beta, -b/k);\end{array}\ z\right] = 1 + \frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=1}^{k}\frac{\Gamma(\alpha-\frac{a}{k}r)}{\Gamma(\beta-\frac{b}{k}r)}\ \frac{z^r}{r!}
$$

$$
\cdot\,{}_{b+1}F_{a+k}\left[\begin{array}{ll}1, \frac{r}{k}+\frac{1-\beta}{b}, \ldots, \frac{r}{k}+\frac{b-\beta}{b} & ;\ \frac{(-b)^b}{(-a)^a}\left(\frac{z}{k}\right)^k\\ \frac{r}{k}+\frac{1-\alpha}{a}, \ldots, \frac{r}{k}+\frac{a-\alpha}{a}, \frac{r+1}{k}, \ldots, \frac{r+k}{k} & ;\end{array}\right] \tag{3.5}
$$

$$
{}_1\Psi_1^*\left[\begin{array}{ll}(\alpha, -a/k) & ;\\ (\beta, b/k) & ;\end{array}\ z\right] = 1 + \frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=1}^{k}\frac{\Gamma(\alpha-\frac{a}{k}r)}{\Gamma(\beta+\frac{b}{k}r)}\ \frac{z^r}{r!}
$$

$$
\cdot\,{}_1F_{a+b+k}\left[1; \frac{r}{k}+\frac{1-\alpha}{a}, \ldots, \frac{r}{k}+\frac{a-\alpha}{a}, \frac{r}{k}+\frac{\beta}{b}, \ldots, \frac{r}{k}+\frac{\beta+b-1}{b}, \frac{r+1}{k}, \ldots, \frac{r+k}{k}; \frac{(z/k)^k}{(-a)^a b^b}\right] \tag{3.6}
$$

$$
{}_1\Psi_1^*\left[\begin{array}{ll}(\alpha, a/k) & ;\\ (\beta, -b/k) & ;\end{array}\ z\right] = 1 + \frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=1}^{k}\frac{\Gamma(\alpha+\frac{a}{k}r)}{\Gamma(\beta-\frac{b}{k}r)}\ \frac{z^r}{r!}
$$

$$
\cdot\,{}_{a+b+1}F_k\left[\begin{array}{ll}1, \frac{r}{k}+\frac{\alpha}{a}, \ldots, \frac{r}{k}+\frac{\alpha+a-1}{a}, \frac{r}{k}+\frac{1-\beta}{b}, \ldots, \frac{r}{k}+\frac{b-\beta}{b} & ;\ a^a(-b)^b(z/k)^k\\ \frac{r+1}{k}, \ldots, \frac{r+k}{k} & ;\end{array}\right]. \tag{3.7}
$$

We mention here that obviously results analogous to Equations (2.4), (3.3) etc. may be obtained for ${}_p\Psi_q^*[z]$ defined by Equation (1.1) when the A_i $(i = 1, \ldots, p)$ and B_i $(i = 1, \ldots, q)$ are integers or rational numbers.

An interesting corollary may be obtained from either Equation (3.3) or (3.5) by setting $a = b$, $\alpha = \beta$; thus

$$\exp z = 1 + \sum_{r=1}^{k} \frac{z^r}{r!}\ {}_1F_k\left[1; \frac{r+1}{k}, \frac{r+2}{k}, \ldots, \frac{r+k}{k}\ ; (z/k)^k\right]$$

where k is a positive integer.

We conclude this section by noting that when $k = 1$, Equations (3.3), (3.5)-(3.7) respectively reduce after simplification to Equations (2.4), (2.6)-(2.8).

4. APPLICATION TO TRINOMIAL EQUATIONS

Interest in the solution of algebraic trinomial equations originated evidently with Lambert (1758). Numerous other investigators studied them, notably Lagrange (1770), Heymann (1887), Capelli (1892), Ramanujan (circa 1903 and 1913), and Mellin (1915). Berndt [6, p. 72, p. 307] gives a brief history of the subject while Belardinelli [7, p. 30] presents a detailed account including an extensive bibliography.

Mellin's result [8] (see also [7, p. 37] and [9, p. 81]) may be given elegantly by employing the Psi function as follows: for real x the positive root of the trinomial equation

$$y^N + xy^{N-Q} - 1 = 0 \ , \qquad N > Q > 0 \tag{4.1}$$

is given by

$$y = {}_1\Psi_1^* \left[\begin{array}{ll} (\frac{1}{N}, \frac{N-Q}{N}) & ; \\ (1+\frac{1}{N}, \frac{-Q}{N}) & ; \end{array} -x \right] \tag{4.2}$$

where Q and N are real numbers and

$$|x| < (Q/N)^{-Q/N}(1 - Q/N)^{Q/N-1} \leq 2 \ . \tag{4.3}$$

Belardinelli observed [7, p. 56] that when N and Q are integers, then the solution y given by Equation (4.2) may be written as a sum of N generalized hypergeometric functions defined by Equation (1.2). However his result is incorrect. Such a representation in terms of generalized hypergeometric functions may be obtained immediately from Equation (3.7) by setting $\alpha = 1/N$, $\beta = 1 + 1/N$, $a = N - Q$, $b = Q$, $k = N$, and $z = -x$. Thus we obtain for positive integers $N > Q \geq 1$ that the positive root of Equation (4.1) is given by

$$y = 1 + \frac{1}{N} \sum_{r=1}^{N} \frac{\Gamma(\frac{1}{N} + \frac{N-Q}{N} r)}{\Gamma(1 + \frac{1}{N} - \frac{Q}{N} r)} \ \frac{(-x)^r}{r!}$$

$$\cdot {}_{N+1}F_N \left[\begin{array}{ll} 1, \frac{r}{N} + \frac{1}{N(N-Q)}, \ldots, \frac{r}{N} + \frac{1}{N(N-Q)} + \frac{N-Q-1}{N-Q}, \frac{r}{N} - \frac{1}{NQ}, \ldots, \frac{r}{N} - \frac{1}{NQ} + \frac{Q-1}{Q} & ; \\ \frac{r+1}{N}, \ldots, \frac{r+N}{N} & ; \end{array} \xi \right] \tag{4.4}$$

where

$$\xi = (-Q)^Q (N-Q)^{N-Q} (-x/N)^N \ . \tag{4.5}$$

We note that since ξ must satisfy $|\xi| < 1$ in order for each ${}_{N+1}F_N[\xi]$ in Equation (4.4) to converge, then Equation (4.5) yields the inequality (4.3). In particular, if $|x| \leq 1$, then we always have $|\xi| < 1$ which will prove useful in section 6.

We remark that it is easy to show that the positive solution y of Equation (4.1) is also given by

$$y^{-1} = 1 + \frac{1}{N} \sum_{r=1}^{N} \frac{,\,(\frac{1}{N} + \frac{Q}{N} r)}{,\,(1 + \frac{1}{N} \; - \; \frac{N-Q}{N} r)} \, \frac{x^r}{r!}$$

$$\cdot {}_{N+1}F_N \left[\begin{array}{ll} 1, \frac{r}{N} + \frac{1}{NQ}, \ldots, \frac{r}{N} + \frac{1}{NQ} + \frac{Q-1}{Q}, \frac{r}{N} \; - \; \frac{1}{N(N-Q)}, \ldots, \frac{r}{N} \; - \; \frac{1}{N(N-Q)} + \frac{N-Q-1}{N-Q} & ; \\ \frac{r+1}{N}, \ldots, \frac{r+N}{N} & ; \end{array} \eta \right]$$

where

$$\eta = (-Q)^Q (N-Q)^{N-Q} (x/N)^N \ . \tag{4.6}$$

In addition, y^{-1} is the positive root of $z^N - xz^Q - 1 = 0$.

Other representations for the positive root of Equation (4.1) (with Q replaced by $N-Q$) may be given. For by using [2, Eqs. (9) and (10)] the trinomial equation

$$y^{N/Q} + xy - 1 = 0$$

for integers $N > Q \geq 1$ has the positive solution

$$y = {}_1\Psi_1^* \left[\begin{array}{ll} (Q/N, Q/N) & ; \\ (1 + Q/N, -1 + Q/N) & ; \end{array} -x \right]$$

where

$$|x| < \frac{N}{Q} \left(\frac{N}{Q} \; - 1 \right)^{\frac{Q}{N} \; -1} \ ; \tag{4.7}$$

thus it is easy to see that

$$y^N + xy^Q - 1 = 0$$

has the solution

$$y = \left({}_1\Psi_1^* \left[\begin{array}{ll} (Q/N, Q/N) & ; \\ (1 + Q/N, -1 + Q/N) & ; \end{array} -x \right] \right)^{1/Q} .$$

Note that y^{-1} is also the positive root of $z^N - xz^{N-Q} - 1 = 0$.

Further, a computation similar to that employed in deriving Equations (3.3) and (3.5)-(3.7), but more complex in its details (so that for brevity we shall omit it here), yields the result:

$$
{}_1\Psi_1^*\left[\begin{array}{ll}(Q/N, Q/N) & ; \\ (1+Q/N, -1+Q/N) & ;\end{array} -x\right] = \frac{(-x)^Q}{N}\delta
$$

$$
+(-1)^Q \frac{Q}{N}\sum_{r=1}^{N-1} \frac{(1-Q+\frac{Q}{N}r)_{Q-1}(\frac{-Q}{N}r)_{r-1}}{(\frac{-Q}{N}r)_Q}\ \frac{x^{r-1}}{(r-1)!}
$$

$$
\cdot {}_N F_{N-1}\left[\begin{array}{ll}1, \frac{r}{N} - \frac{1}{N-Q}, \frac{r}{N}, \ldots, \frac{r}{N} + \frac{N-Q-2}{N-Q}, \frac{r}{N} + \frac{1}{Q}, \ldots, \frac{r}{N} + \frac{Q-1}{Q} & ; \\ \frac{r+1}{N}, \ldots, \frac{r+N-1}{N} & ;\end{array} \eta\right] \qquad (4.8)
$$

where η is given by Equation (4.6), δ is defined by

$$
\delta = \begin{cases} 0 & N \neq Q+1 \\ 1 & N = Q+1 \end{cases}
$$

and x satisfies the inequality (4.7).

In particular, setting $Q = 1$ in Equation (4.8) we see that the positive root of the equation

$$
y^N + xy - 1 = 0\ , \qquad N > 2
$$

is given by

$$
y = \sum_{r=1}^{N-1}\left(\frac{-r}{N}\right)_{r-1}\frac{x^{r-1}}{r!}\,{}_N F_{N-1}\left[\begin{array}{ll}1, \frac{r}{N} - \frac{1}{N-1}, \frac{r}{N}, \ldots, \frac{r}{N} + \frac{N-3}{N-1} & ; \\ \frac{r+1}{N}, \ldots, \frac{r+N-1}{N} & ;\end{array} -(N-1)^{N-1}(x/N)^N\right].
$$

5. CONNECTION WITH MEIJER'S G-FUNCTION

In [10] Boersma shows for positive integers a_i $(i = 1, \ldots, p)$, b_i $(i = 1, \ldots, q)$, and k that the Fox-Wright Psi function

$$
{}_p\Psi_q^*\left[\begin{array}{ll} (\alpha_1, a_1/k), \ldots, (\alpha_p, a_p/k) & ; \\ (\beta_1, b_1/k), \ldots, (\beta_q, b_q/k) & ; \end{array} z\right] = \mu G^{k,a}_{a,k+b}\left(\left(\frac{-z}{k}\right)^k \frac{a_1^{a_1}\ldots a_p^{a_p}}{b_1^{b_1}\ldots b_q^{b_q}} \,\middle|\, \begin{array}{l} \xi_1, \ldots, \xi_a \\ \eta_1, \ldots, \eta_{k+b} \end{array}\right)
$$

where

$$
a = \sum_{i=1}^{p} a_i \ , \qquad b = \sum_{i=1}^{q} b_i
$$

$$
\mu = \frac{,(\beta_1)\ldots,(\beta_q)}{,(\alpha_1)\ldots,(\alpha_p)}\,\frac{\sqrt{k}}{2\pi^{\frac{k-1}{2}}}\,\frac{\prod_{i=1}^{q}(2\pi)^{\frac{b_i-1}{2}} b_i^{\frac{1}{2}-\beta_i}}{\prod_{i=1}^{p}(2\pi)^{\frac{a_i-1}{2}} a_i^{\frac{1}{2}-\alpha_i}}
$$

and

$$
\xi_1 = 1 - \frac{\alpha_1}{a_1}\ , \quad \xi_2 = 1 - \frac{\alpha_1 + 1}{a_1}, \ldots, \xi_{a_1} = 1 - \frac{\alpha_1 + a_1 - 1}{a_1}
$$

$$
\xi_{a_1+1} = 1 - \frac{\alpha_2}{a_2}\ , \quad \xi_{a_1+2} = 1 - \frac{\alpha_2 + 1}{a_2}, \ldots, \xi_{a_1+a_2} = 1 - \frac{\alpha_2 + a_2 - 1}{a_2}
$$

$\ldots$

$$
\xi_{a_1+a_2+\ldots+a_{p-1}+1} = 1 - \frac{\alpha_p}{a_p}\ , \quad \xi_{a_1+a_2+\ldots+a_{p-1}+2} = 1 - \frac{\alpha_p + 1}{a_p}, \ldots, \xi_a = 1 - \frac{\alpha_p + a_p - 1}{a_p};
$$

$$
\eta_1 = 0\ , \qquad \eta_2 = \frac{1}{k}, \ldots, \eta_k = \frac{k-1}{k}
$$

$$
\eta_{k+1} = 1 - \frac{\beta_1}{b_1}\ , \qquad \eta_{k+2} = 1 - \frac{\beta_1 + 1}{b_1}, \ldots, \eta_{k+b_1} = 1 - \frac{\beta_1 + b_1 - 1}{b_1}
$$

$$
\eta_{k+b_1+1} = 1 - \frac{\beta_2}{b_2}\ , \qquad \eta_{k+b_1+2} = 1 - \frac{\beta_2 + 1}{b_2}, \ldots, \eta_{k+b_1+b_2} = 1 - \frac{\beta_2 + b_2 - 1}{b_2}
$$

$\ldots$

$$
\eta_{k+b_1+b_2+\ldots+b_{q-1}+1} = 1 - \frac{\beta_q}{b_q}\ , \quad \eta_{k+b_1+b_2+\ldots+b_{q-1}+2} = 1 - \frac{\beta_q + 1}{b_q}, \ldots, \eta_{k+b} = 1 - \frac{\beta_q + b_q - 1}{b_q}\ .
$$

In particular, for $p = q = 1$ we get

$$
{}_1\Psi_1^*\left[\begin{array}{ll} (\alpha, a/k) & ; \\ & z \\ (\beta, b/k) & ; \end{array}\right] = 2\pi^{\frac{1+b-a-k}{2}}\sqrt{\frac{kb}{a}}\,\frac{\Gamma(\beta)}{\Gamma(\alpha)}\,\frac{a^\alpha}{b^\beta}
$$

$$
\cdot G^{k,a}_{a,k+b}\left(\left(\frac{-z}{k}\right)^k \frac{a^a}{b^b}\,\middle|\, \begin{array}{l} 1-\frac{\alpha}{a},\ldots,1-\frac{\alpha+a-1}{a} \\ 0,\frac{1}{k},\ldots,\frac{k-1}{k},1-\frac{\beta}{b},\ldots,1-\frac{\beta+b-1}{b} \end{array}\right). \tag{5.1}
$$

Now comparing the latter result with Equation (3.3) we obtain for positive integers a, b, and k:

$$
G^{k,a}_{a,k+b}\left(\left(\frac{-z}{k}\right)^k \frac{a^a}{b^b}\,\middle|\, \begin{array}{l} 1-\frac{\alpha}{a},\ldots,1-\frac{\alpha+a-1}{a} \\ 0,\frac{1}{k},\ldots,\frac{k-1}{k},1-\frac{\beta}{b},\ldots,1-\frac{\beta+b-1}{b} \end{array}\right) = 2\pi^{\frac{a-b+k-1}{2}}\sqrt{\frac{a}{kb}}\,\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,\frac{b^\beta}{a^\alpha}
$$

$$
\cdot\left(1+\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=1}^{k}\frac{\Gamma(\alpha+\frac{a}{k}r)}{\Gamma(\beta+\frac{b}{k}r)}\,\frac{z^r}{r!}\;{}_{a+1}F_{b+k}\left[\begin{array}{ll} 1,\frac{r}{k}+\frac{\alpha}{a},\ldots,\frac{r}{k}+\frac{\alpha+a-1}{a} & ; \\ \frac{r}{k}+\frac{\beta}{b},\ldots,\frac{r}{k}+\frac{\beta+b-1}{b},\frac{r+1}{k},\ldots,\frac{r+k}{k} & ; \end{array}\left(\frac{z}{k}\right)^k\frac{a^a}{b^b}\right]\right). \tag{5.2}
$$

See [11, Ch. V], for example, for an introduction to the G-function.

6. A PROBLEM IN INFORMATION THEORY

We consider a noiseless and memoryless communication channel [12] with symbols s_1 and s_2. The time for symbol s_1 (s_2) to pass through the channel is the positive integer $Q(N)$, $N > Q$. We note that the case $N = Q$ is trivial and will not be discussed.

A transmission over the channel can be viewed as a sequence s whose terms are s_1 or s_2. We define the length of s to be $c_1Q + c_2N$, where c_i is the number of occurrences of s_i in s. Let S_n, n an integer, be the set consisting of all sequences of length n, and let $|S_n|$ denote the magnitude or cardinal number of S_n. Since there is only one sequence of length zero (the empty sequence), $|S_0| = 1$. There are no sequences of negative length, so $|S_{-|n|}| = 0$, $n \neq 0$.

The maximal amount of "information," in units of bits per unit time, that can be transmitted over the channel is called the capacity C and is defined by

$$C = \limsup_{n\to\infty} \ \log_2 \ \sqrt[n]{|S_n|} \ . \tag{6.1}$$

The original definition was given by Shannon [12, p. 37] who used the ordinary limit, which is not always defined (i.e., $Q = 2$, $N = 4$, $|S_{2n+1}| = 0$). The limit superior is always defined since $|S_n|$ is bounded from above by 2^n. Other authors have noted and corrected Shannon's definition, but the error nevertheless has been perpetuated through much of the literature. However, in practice, the following correct result [12, p. 37] is often used to express C.

Theorem (Shannon): The capacity is given by $C = \log_2 y$, where y is the unique positive root of the equation $z^N - z^{N-Q} - 1 = 0$.

For the same reason that Shannon's definition is flawed, his proof of the theorem, using the asymptotic behavior of finite differences, does not hold. Certainly others have correctly proved the result, but we offer below a novel proof using complex analysis. We note that a similar method has been employed by Kuich [13] in his study of the entropy of context-free languages.

If we place the terms s_1, s_2, respectively, onto the end of nonempty sequences of length $n-Q, n-N$, we obtain in each case a sequence of length n. Since the last term of a nonempty sequence of length n must be either s_1 or s_2, we see that in general the $|S_n|$ satisfy the recurrence relation

$$|S_n| = |S_{n-Q}| + |S_{n-N}| + \delta_{0n}$$

where $\delta_{0n} = 1$ if and only if $n = 0$.

We define the z-transform [14] of a sequence $\{a_n\}$ to be the power series $\sum_{n=0}^{\infty} a_n z^n$. Applying the z-transform to the above, we arrive at the formal equation

$$\sum_{n=0}^{\infty} |S_n| z^n = z^N \sum_{n=0}^{\infty} |S_n| z^n + z^Q \sum_{n=0}^{\infty} |S_n| z^n + 1 \ .$$

The above result is valid for $|z|$ less than the radius of convergence of the power series $S(z) = \sum_{n=0}^{\infty} |S_n| z^n$ and so we have

$$S(z) = \frac{1}{1 - z^Q - z^N} \ .$$

Since $|S_n| \leq 2^n$, $S(z)$ is analytic in a neighborhood of $z = 0$ and so may be expressed uniquely as a Maclaurin series. Therefore, the radius of convergence of $S(z)$, which is given by $1/\limsup_{n\to\infty} \sqrt[n]{|S_n|}$, is equal to the magnitude of the root of smallest modulus of $1 - z^Q - z^N = 0$.

Now noting that

$$1 - |z|^Q - |z|^N \leq |1 - z^Q - z^N|$$

we see that the positive root r of $1 - z^Q - z^N = 0$ is the root of smallest modulus; see also [15, p. 122, Theorem (27,1)]. Thus we have

$$r = 1/\limsup_{n\to\infty} \sqrt[n]{|S_n|} \ ;$$

and so

$$y = r^{-1} = \limsup_{n\to\infty} \sqrt[n]{|S_n|}$$

is the positive root of $z^N - z^{N-Q} - 1 = 0$.

Next, by Equation (6.1)

$$\begin{aligned} C &= \limsup_{n\to\infty} \ \log_2 \ \sqrt[n]{|S_n|} \\ &= \log_2 \ \limsup_{n\to\infty} \ \sqrt[n]{|S_n|} \\ &= \log_2 y \end{aligned}$$

and the theorem is proved.

Thus by setting $x = -1$, for example, in Equation (4.4) we have the following

Corollary: For $N > Q \geq 1$ the capacity $C = C(N, Q)$ is given by

$$2^C = 1 + \frac{1}{N} \sum_{r=1}^{N} \frac{,\,(\frac{1}{N} + \frac{N-Q}{N} r)}{,\,(1 + \frac{1}{N} \ - \ \frac{Q}{N} r) r!}$$

$$\cdot \ {}_{N+1}F_N \left[\begin{array}{ll} 1, \mu(r), \ldots, \mu(r) + \frac{N-Q-1}{N-Q}, \nu(r), \ldots, \nu(r) + \frac{Q-1}{Q} & ; \\ \frac{r+1}{N}, \ldots, \frac{r+N}{N} & ; \end{array} \xi \right]$$

where

$$\mu(r) = \frac{r}{N} + \frac{1}{N(N-Q)} \ , \qquad \nu(r) = \frac{r}{N} \ - \ \frac{1}{NQ}$$

and

$$\xi = (-Q)^Q (N-Q)^{N-Q} / N^N \ .$$

Closed forms for the capacity for $1 \leq Q < N \leq 4$ are of course readily given since polynomial equations up to the fourth degree are "solvable by radicals." Thus we have, for example, the following:

$$2^{C(2,1)} = \frac{1 + \sqrt{5}}{2}$$

$$2^{C(3,1)} = \frac{1}{3} \left[1 + \left(\frac{\sqrt{31} \ - \ \sqrt{27}}{2} \right)^{2/3} + \left(\frac{\sqrt{31} \ + \ \sqrt{27}}{2} \right)^{2/3} \right]$$

$$2^{C(3,2)} = \frac{\sqrt{3}}{3} \left[1 + \left(\frac{\sqrt{27} \ - \ \sqrt{23}}{2} \right)^{1/3} + \left(\frac{\sqrt{27} \ + \ \sqrt{23}}{2} \right)^{1/3} \right]$$

$$2^{C(4,2)} = \frac{1}{2} \left(2 \ + \ \sqrt{20} \right)^{1/2} \ .$$

In general, however, $2^{C(N,Q)}$ may be expressed by means of the results previously given in terms of a finite number P of hypergeometric functions ${}_{P+1}F_P[z]$ where $P = N$ or $N-1$. In [16] and [17] extensive and up to date collections of reductions for single, double, and multiple generalized hypergeometric functions are provided.

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